

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 3

Daily Tutorial Sheet - 13

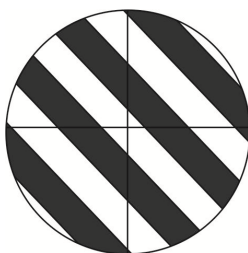
- 156.** The arithmetic-geometric means inequality gives $\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} \geq 2(\sin x \cos x)^{y-y^2/4}$.

It follows that $2 \sin x \cos x = \sin 2x \geq (\sin x \cos x)^{y-y^2/4}$, and because $\sin x \cos x < 1$, it follows that $1 \leq y - y^2/4$, or $(1 - y/2)^2 \leq 0$.

It follows that all the equalities hold; that is, $y = 2$ and $\sin x = \cos x$, and so there is a unique solution:

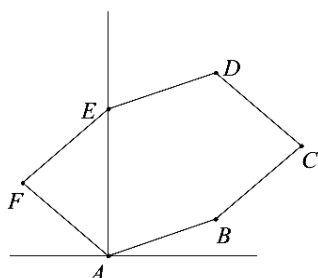
$$(x, y) = \left(\frac{\pi}{4}, 2 \right).$$

- 157.** Let c denote the disk (x, y) with $x^2 + y^2 \leq 100$. Because $\sin(x + y) = 0$ if and only if $x + y = k\pi$ for integers k , disk c has been cut by parallel lines $x + y = k\pi$, and in between those lines there are regions containing points (x, y) with either $\sin(x + y) > 0$ or $\sin(x + y) < 0$. Since $\sin(-x - y) = -\sin(x + y)$, the regions containing points (x, y) with $\sin(x + y) > 0$ are symmetric with respect to the origin to the regions containing points (x, y) with $\sin(x + y) < 0$. Thus, as indicated in figure, the area of region $\Re$ is half the area of disc c , that is, 50π



- 158.** Without loss of generality, we assume that $b > 0$. (Otherwise, we can reflect the hexagon across the y -axis.) Let the x coordinates of C, D, E and F be c, d, e and f , respectively. Note that the y coordinate of C is not 4, since if it were, the fact $|AB| = |BC|$ would imply that A, B and C are collinear or that $c = 0$, implying that $A B C D E F$ is concave. Therefore, $F = (f, 4)$. Since $\overline{AF} = \overline{CD}$, $C = (c, 6)$ and $D = (d, 10)$ and so $E = (e, 8)$. Because the y coordinates of B, C and D are 2, 6 and 10, respectively and $|BC| = |CD|$, we conclude that $b = d$. Since $\overline{AB} = \overline{ED}$, $e = 0$. Let a denote the side length of the hexagon. Then $f < 0$. We need to compute $[A B C D E F] = [A B D E] + [A E F] + [B C D] = [A B D E] + 2[A E F]$

$$= b \cdot A E + (-f) \cdot A E = 8(b - f)$$



Let α denote the measure (in degrees) of the standard angle formed by the line AB and the x axis. Then the standard angle formed by the line AF and the x axis is $\beta = 120^\circ + \alpha$. By considering the y coordinates of B and F, we have $a \sin \alpha = 2$ and $4 = a \sin(120^\circ + \alpha) = \frac{a\sqrt{3} \cos \alpha}{2} - \frac{a \sin \alpha}{2} = \frac{a\sqrt{3} \cos \alpha}{2} - 1$, by the addition and subtraction formulas. Hence $a \cos \alpha = \frac{10}{\sqrt{3}}$. Thus, by considering the x coordinates of B and F, we have $b = a \cos \alpha = \frac{10}{\sqrt{3}}$ and $f = a \cos(120^\circ + \alpha) = -\frac{a \cos \alpha}{2} - \frac{a\sqrt{3} \sin \alpha}{2} = -\frac{8}{\sqrt{3}}$

- 159.** Set $a = \tan^2 \alpha$ and $b = \tan^2 \beta$. It suffices to determine the minimum value of $\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a}$, with $a, b \geq 0$.

We have $\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} = \frac{a^2 + 2a + 1}{b} + \frac{b^2 + 2b + 1}{a} = \left(\frac{a^2}{b} + \frac{1}{b} + \frac{b^2}{a} + \frac{1}{a} \right) + 2 \left(\frac{a}{b} + \frac{b}{a} \right)$
 $\geq 4 \sqrt{\frac{a^2}{b} \cdot \frac{1}{b} \cdot \frac{b^2}{a} \cdot \frac{1}{a}} + 4 \sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 8$, by the arithmetic-geometric means inequality. Equality holds when $a = b = 1$; that is, $\alpha = \beta = n\pi \pm \frac{\pi}{4}$ for $k \in \mathbb{Z}$

- 160.** If $\cos x = 0$, the desired inequality reduces to $\sin^2 x \leq 1 + \left(\frac{a+b}{2} \right)^2$, which is clearly true. We assume that $\cos x \neq 0$. Dividing both sides of the desired inequality by $\cos^2 x$ gives

$$(\tan x + a)(\tan x + b) \leq \left[1 + \left(\frac{a+b}{2} \right)^2 \right] \sec^2 x.$$

Let $t = \tan x$. Then $\sec^2 x = 1 + t^2$.

The above inequality reduces to $t^2 + (a+b)t + ab \leq \left(\frac{a+b}{2} \right)^2 t^2 + t^2 + \left(\frac{a+b}{2} \right)^2 + 1$

$$\text{or } \left(\frac{a+b}{2} \right)^2 t^2 + 1 - (a+b)t + \left(\frac{a+b}{2} \right)^2 - ab \geq 0$$

The last inequality is equivalent to $\left(\frac{(a+b)t}{2} - 1 \right)^2 + \left(\frac{a-b}{2} \right)^2 \geq 0$, which is evident.

- 161.** Rewrite the two given equations as:

$$\sin a - 8 \sin d = 4 \sin c - 7 \sin b \quad \text{and} \quad \cos a - 8 \cos d = 4 \cos c - 7 \cos b.$$

By squaring the two equations and adding them, we obtain

$$1 + 64 - 16(\cos a \cos d + \sin a \sin d) = 16 + 49 - 56(\cos b \cos c + \sin b \sin c) = 2 \cos(a-d) = 7 \cos(b-c)$$